Quadratic counts of rational curves on a quintic 3-fold

using Marc Levine's quadratic version of Bott's residue formula

Conjecture (Clemens 1984): $X = \{ F = 0 \} \subset \mathbb{P}^4$, F degree 5

A general quintic hypersurface in $\mathbb{P}^4$ contains only finitely many smooth rational curves of degree d.

true for $d \leq 11$, unknown else

How many?

$d=1$ (lines): 2875

$d=2$ (conics): 609 250 Katz 1986

$d=3$ (twisted cubics): 317 206 375 Ellingsrud - Strømme 1994

formula for general d by Candelas - de la Ossa - Green (1991)

proved by Givental (1996), Lian - Liu - Yau (1997)

How to count rational curves of degree $d \leq 3$ on a quintic 3-fold

$$\begin{array}{c} V \\ \downarrow \uparrow \sigma_F \\ \mathcal{M} \end{array}$$

Vector bundle V

moduli space of smooth rational curves of degree d in $\mathbb{P}^4$ $\to \mathcal{M}$

e.g. $d=1$ $G(1,4)$

$X = \{F = 0\} \subseteq \mathbb{P}^4$ defines a section σ_F of $V \to \mathcal{M}$

degree 5 (of F)

rational degree d curves on X $=$ # zeros of σ_F $=$ degree $e(V)$ (Euler class)

Goal: Find an efficient method to compute $\deg e(V)$.

Equivariant cohomology

$G \curvearrowright X$

- G algebraic group scheme $/\mathbb{C}$
- X algebraic variety $/\mathbb{C}$

$$EG \longrightarrow BG = EG/G$$

contractible with free G-action

classifying space

$$X_G := \frac{EG \times X}{(e \cdot g, x) \sim (e, g \cdot x)}$$

$$H_G^*(X) := H^*(X_G)$$

Example:

- $G = \mathbb{C}^*$

$$H_G^*(pt) = H^*(BG) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[t]$$

- $G = (\mathbb{C}^*)^n$

$$H_G^*(pt) = \mathbb{Z}[t_1, \dots, t_n]$$

For $V \to X$ G-vector bundle

$\rightsquigarrow$ vector bundle $V_G \longrightarrow X_G$ $\quad e_G(V) := e(V_G)$

Theorem (Bott's residue formula) $G = (\mathbb{C}^*)^s$

$G \curvearrowright X$ with finitely many fixed pts $p_1, \dots, p_n$, $\alpha \in H_G^*(X)$

smooth proper variety/$\mathbb{C}$

$$\boxed{\deg \alpha = \sum_{j=1}^{n} \deg \frac{i_{p_j}^* \alpha}{e_G(T_{p_j} X)}}$$

$$\alpha \in H_G^*(X) \xrightarrow{\pi_{X*}} H_G^*(pt) = H^*(BG) = \mathbb{Z}[t_1, \dots, t_s]$$

$\pi_X : X \to \{pt\}$

$\deg \alpha := \pi_{X*} \alpha \in H^*(BG)$

$i_{p_j}^* \alpha \in H_G^*(\{p_j\})$

$i_{p_j} : \{p_j\} \hookrightarrow X$

Bott's residue formula

$$\boxed{\deg \alpha = \sum_{j=1}^{n} \deg \frac{i_{p_j}^* \alpha}{e_G(T_{p_j} X)}}$$

<u>Goal</u>: Compute $\deg(e(V))$

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "\pi_X"'] & X_G \arrow[d, "\pi_{X_G}"] \\ pt \arrow[r, "i_{pt}"] & BG \end{tikzcd} \rightsquigarrow \begin{tikzcd} e(V) \in H^*(X) \arrow[d, "\pi_{X*}"'] & H_G^*(X) \ni e_G(V) \arrow[l, "i_X^*"'] \arrow[d, "\pi_{X_G *}"] \\ \deg e(V) \in H^*(pt) & H^*(BG) \ni \deg e_G(V) \arrow[l, "i_{pt}^*"'] \end{tikzcd}$$

<u>Example</u>: $\chi(\mathbb{CP}^n) = \deg(e(T\mathbb{CP}^n)) = \deg e_G(T\mathbb{CP}^n)$, $G = (\mathbb{C}^*)^{n+1}$

$(\mathbb{C}^*)^{n+1} \curvearrowright \mathbb{CP}^n$ with fixed pts of the form $[0:\dots:0:1:0\dots:0]$

$$\deg e_G(T\mathbb{CP}^n) = \sum_p \deg \frac{e_G(\cancel{T_p\mathbb{CP}^n})}{e_G(\cancel{T_p\mathbb{CP}^n})} = \#\text{ fixed pts} = n+1$$

Example: # lines on a quintic 3-fold $= \deg e(\mathrm{Sym}^5 S^\vee \to G(1,4))$

$G = (\mathbb{C}^*)^5 \curvearrowright \mathbb{CP}^4 \rightsquigarrow (\mathbb{C}^*)^5 \curvearrowright G(1,4)$

has $\binom{5}{3} = 10$ fixed pts $\{x_i = x_j = x_\ell = 0\}$

$H^*(BG) = \mathbb{Z}[t_0, \dots, t_4]$

$$\sum_{\substack{\text{fixed pts} \\ p}} \deg \left(\frac{e_G(\mathrm{Sym}^5 S_p^\vee)}{e_G(T_p G(1,4))} \right| = 2875$$

Easier: # lines on a cubic surface $= 27$

Example: # rational curves of degree $d \leq 3$ on a quintic 3-fold

$$= \deg e_G(V) = \begin{cases} 609\,250 & d=2 \\ 317\,206\,375 & d=3 \end{cases}$$ (Ellingsrud - Strømme)

Grothendieck-Witt ring $GW(k)$ of a field k $\quad$ char $k \neq 2$

$GW(k) :=$ group completion of semi-ring of isometry classes of non-degenerate quadratic forms $/k$

$q_1 : V_1 \to k$

$q_2 : V_2 \to k$

$q_1 \oplus q_2 : V_1 \oplus V_2 \to k$

$\otimes \qquad \otimes$

generators $\quad \langle a \rangle \overset{= \langle ab^2 \rangle}{=} ax^2 \overset{"=" ab^2x^2}{} \quad$ for $\quad a \in k^\times/(k^\times)^2$

relations $\quad$ 1) $\langle a \rangle \langle b \rangle = \langle ab \rangle$

2) $\langle a \rangle + \langle b \rangle = \langle a+b \rangle + \langle ab(a+b) \rangle$

3) $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle =: h \quad$ hyperbolic form

Witt ring of k

$$W(k) := \frac{GW(k)}{\langle h \rangle} = \frac{GW(k)}{\mathbb{Z} \cdot h}$$

$q(x_1, \dots, x_n)$

$= a_1 x_1^2 + a_2 x_2^2 + \dots + a_n x_n^2$

Example: $\quad \bullet \; GW(\mathbb{C}) \cong \mathbb{Z} \qquad W(\mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z}$

$\bullet \; GW(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z} \qquad W(\mathbb{R}) \cong \mathbb{Z}$

Enumerative geometry = count numbers of solutions to geometric questions

Example: Count of rational curves of degree d on a quintic 3-fold.

- count does <u>not</u> depend on choice of quintic 3-fold when counting over $k = \bar{k}$
- this invariance breaks down over non-alg closed fields

$\mathbb{A}^1$-enumerative geometry = get invariant count over arbitrary k when "counting" in $GW(k)$ or $W(k)$

Example: (M. Levine, P.)

Quadratic count of lines on a quintic 3-fold

$1430 \cdot h + 15 \cdot \langle 1 \rangle \in GW(k)$ or $15 \langle 1 \rangle \in W(k)$

in $GW(\mathbb{C}) \cong \mathbb{Z}$ this is $1430 \cdot 2 + 15 = 2875$

Cohomology with Witt valued coefficients

- $\mathcal{W}$ sheaf on Sm_k ← cat of smooth varieties $/k$
- can take cohomology with coefficients in this sheaf

  eg $H^*(\operatorname{Spec} k, \mathcal{W}) = W(k)$
- have pullback
- proper pushforward with twists

$$H^*(X, \mathcal{W}(\omega_{X/k})) \to H^*(Y, \mathcal{W}(\omega_{Y/k}))$$

- have equivariant cohomology (A. d'Angelo)

Theorem (Ananyevskiy):

$$\boxed{H^*(BSL_2, \mathcal{W}) = W(k)[t]}$$

Recall:

$$H^*(B\mathbb{C}^*) = \mathbb{Z}[t]$$

$$\cancel{H^*(B\mathbb{C}^*, \mathcal{W}) = W(k)}$$

Bott's residue formula for cohomology with Witt coeff

Thm (M. Levine)

1) $G = (SL_2)^n \curvearrowright X$ for X "nice" $\in$ variety$/k$

with finitely many fixed pts

2) $G = N$ = normalizer of $t \in \mathbb{G}_m$ in SL_2^n $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$

$G \curvearrowright X$ with finitely many fixed pts

Then for $\alpha \in H^*_G(X, \mathcal{W}(\omega_{X/k}))$

$$\boxed{\deg \alpha = \sum_{p \text{ fixed pt}} \deg \frac{i_p^* \alpha}{e_G(T_p X)}}$$

Quadratic count of rational curves of degree $d \leq 3$

$\underline{d=1}$: $\quad G = (SL_2)^2 \circlearrowright \mathbb{P}^4$

$\rightsquigarrow \quad G \circlearrowright \mathbb{G}(1,4)$ has 2 fixed pts $\ell_1 = \{x_2 = x_3 = x_4 = 0\}$

and $\ell_2 = \{x_0 = x_1 = x_4 = 0\}$

$$e(\mathrm{Sym}^5 S^\vee) \in H^*(\mathbb{G}(1,4), W(\det{}^{-1} \mathrm{Sym}^5 S^\vee))$$

$$\cong$$

$$H^*(\mathbb{G}(1,4), W(\omega_{\mathbb{G}(1,4)/k}))$$

"relatively orientable"

$$\deg e_G(\mathrm{Sym}^5 S^\vee) = \deg \frac{e_G(\mathrm{Sym}^5 S^\vee_{\ell_1})}{e_G(T_{\ell_1}\mathbb{G}(1,4))} + \deg \frac{e_G(\mathrm{Sym}^5 S^\vee_{\ell_2})}{e_G(T_{\ell_2}\mathbb{G}(1,4))}$$

$$\rotatebox{90}{$\in$}$$

$$H^*(B(SL_2)^2, W)$$

$$\|$$

$$W(k)[t_1, t_2]$$

$$= 15 \cdot \langle 1 \rangle \frac{t_1^3}{t_1(t_1^2 - t_2^2)} + 15 \langle 1 \rangle \frac{t_2^3}{t_2(t_2^2 - t_1^2)} = 15 \langle 1 \rangle \in W(k)$$

$\underline{d = 2,3:}$ $N \curvearrowright \mathcal{M}$

with finitely many fixed pts

$$e(V) \in H^*(\mathcal{M}, \mathcal{W}(\det^{-1} V))$$

$$\cong_2$$

$$H^*(\mathcal{M}, \mathcal{W}(\det_{\mathcal{M}/k}))$$

$$\deg e_N(V) = \begin{cases} 0 & d = 2 \\ 765 \cdot \langle 1 \rangle & d = 3 \end{cases} \qquad (\text{Levine} - \text{P.})$$

$V_{\mathbb{C}} = \mathbb{R}$ $\langle 1 \rangle$, $\langle -1 \rangle$

317 206 375

THANK

YOU!